ABSTRACT: In this presentation, the basic theoretical background of the Hilbert Transform is introduced. Using this transform, normal real-valued time domain functions are made complex. This yields two useful properties - the Envelope and the Instantaneous Frequency. Examples of the practical use of these functions are demonstrated, with emphasis on acoustical applications.
Overview of topics covered.
Hilbert Transform
What is it?

- Time Domain:
  \( \frac{\lambda}{4} \) shift for all frequencies

- Frequency Domain:
  -90° phase shift for all spectral components

The Hilbert Transform does not change domains. A Time Domain Function remains in the Time Domain and a Frequency Domain Function remains in the Frequency Domain. The effect is similar to an integration.
The Hilbert Transform in the Time Domain can be written as a convolution.

\[ H[a(t)] = \tilde{a}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} a(\tau) \frac{1}{t - \tau} \, d\tau \]

\[ = \frac{1}{\pi} a(t) \ast \frac{1}{t} \]
Even & Odd Functions

A causal time signal shown as a sum of an even signal and an odd signal

\[ a(t) = a_e(t) + a_o(t) \]

A few tools to understand the Hilbert Transform with a minimum of mathematics.
## Fourier Transform Relationships

<table>
<thead>
<tr>
<th>$a(t)$</th>
<th>$\mathcal{F}$</th>
<th>$A(f) = R(f) + jX(f)$</th>
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<tbody>
<tr>
<td>Real and Even</td>
<td>$\mathcal{F}$</td>
<td>Real and Even</td>
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<td>Real and Odd</td>
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<td>Imaginary and Odd</td>
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<td>Real</td>
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<td>$R$ Even, $X$ Odd</td>
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<td>Imaginary</td>
<td>$\mathcal{F}$</td>
<td>$R$ Odd, $X$ Even</td>
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From the Symmetry Property of the Fourier Transform: $F\{F\{a(t)\}\} = a(-t)$. 
“Sign” Function

Multiplication by $\text{sgn}(x)$ changes an even function to odd and vice-versa

The Fourier Transform of the $\text{sgn}(x)$ function is $1/X$. 
Using these tools, we can write the Hilbert Transform in the Frequency Domain as shown.
Effect of the Hilbert Transform in the Frequency Domain.
Successive Hilbert Transforms of a sinusoid.
Definition of an Analytic Signal and the Envelope Function. The spectrum of the Analytic signal is one-sided (positive only) and positive valued.
Analytic Signal - Descriptors

Magnitude

\[ |a(t)| = \sqrt{a^2(t) + \tilde{a}^2(t)} \]

Instantaneous Phase

\[ \theta(t) = \tan^{-1} \frac{\tilde{a}(t)}{a(t)} \]

Instantaneous Frequency

\[ f_i(t) = \frac{1}{2\pi} \frac{d\theta(t)}{dt} \]

Additional signal descriptors available from the Hilbert Transform and Analytic Signal.
Analytic Signal - Example

Real  \[ a(t) = \cos 2\pi f_0 t \]

Imaginary  \[ \tilde{a}(t) = \sin 2\pi f_0 t \]

Magnitude \[ |a(t)| = 1 \]

Phase  \[ \theta(t) = 2\pi f_0 t \]

Instantaneous Frequency  \[ f_i(t) = f_0 \]

Example.
Sinusoidal Analytic Signal shown as a “Heyser Spiral”. The Nyquist Plot is the projection along the time axis. The Real and Imaginary parts are the projections along the real and imaginary axes, respectively.
Time & Frequency relationships of the various descriptors for a causal two-port system response function.
Envelope Extraction

Application of the Hilbert Transform to Envelope extraction.
Decay Time Estimation ($\tau, T_{60}$)

Application of the Hilbert Transform to Envelope extraction and decay time estimation, RT60 or RC LC circuit time constant, etc. The positive-valued magnitude function can be graphed on a log amplitude scale, enabling a far wider dynamic range than for a real-valued time signal.
Application of the Hilbert Transform to Envelope to acoustic signal propagation time estimation.
Practical Example

Example: Loudspeaker at 1 metre.
Loudspeaker Measurement

Resulting response without windowing, shown as magnitude in both the Time and Frequency Domains.
For a minimum Phase system, it can be shown that the phase is not independent of the magnitude, but an be derived using the Hilbert Transform as shown. The All-Pass function has poles and zeroes that are negative conjugates of one-another, so the magnitude is unity. The phase of the All-Pass is a pure delay.
MatLab Usage

Y = HILBERT(X) computes the so-called discrete-time analytic signal

\[ Y = \text{Re} \{ X \} + j \cdot \tilde{X} \]

where \( \tilde{X} \) is the Hilbert transform of the vector \( \text{Re} \{ X \} \).

Example: Mute Analysis

The HILBERT function in MATLAB.
Use of the Hilbert Transform:

• Allows definition of the analytic signal from a real-valued time signal:
  – The Real Part is the original time signal
  – The Imaginary Part is the Hilbert Transform

• This enables calculation of the envelope (magnitude) of a time signal

• Applications:
  – Envelope Extraction / Magnitude Estimation
  – Decay Time Estimation: $RT_{60}$, $\tau$
  – Propagation Delay & Signal Arrival Measurements
  – All-Pass / Minimum Phase System Separation

Conclusion.